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Converse of the Eilers–Horst Theorem

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We generalize the theorem of Eilers and Horst, showing that any finite as well as any σ-finite measure on a quantum logic of all closed subspaces of a Hilbert space H of dimension $\neq 2$ is a Gleason one iff the dimension of H is a nonmeasurable cardinal.

1. INTRODUCTION

One of the most important examples of quantum logic is a system $\mathscr{L}(H)$ of all closed subspaces of a (not necessarily separable) real or complex Hilbert space H. A measure on $\mathscr{L}(H)$ is a map $m: \mathscr{L}(H) \to [0, \infty]$ such that (1) $m(0) = 0$; (2) $m(\bigoplus_{n=1}^{\infty} M_n) = \sum_{n=1}^{\infty} m(M_n)$ whenever $M_n \perp M_m$ for $n \neq m$. The famous theorem of Gleason (1957) says that any finite measure m on a separable Hilbert space H, $\dim H \neq 2$, is in a one-to-one correspondence with a positive Hermitian operator T on H of finite trace via

$$m(M) = \mathrm{tr}(TP^M), \qquad M \in \mathscr{L}(H) \tag{1}$$

where P^M is the orthoprojector of H onto M.

Eilers and Horst (1975) proved that the assumption of separability is superfluous when the dimension of the Hilbert space is nonmeasurable cardinal [see also Drisch (1979)]. Recall that, according to Ulam (1930), the cardinal I is nonmeasurable if there is no probability measure ν on the power set 2^A of a set A whose cardinal is I such that $\nu(\{a\}) = 0$ for any $a \in A$.

In the present note we give a new characterization of nonmeasurable cardinals via Gleason measures.

2. EILERS–HORST THEOREM

Let $\mathbf{n}$ be a cardinal. We say that a measure m is (1) $\mathbf{n}$-finite if there is a system $\{M_t: t \in T\}$ of mutually orthogonal subspaces of H such that

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$H = \bigoplus_{t \in T} M_t$, card $T = \mathbf{n}$, and $m(M_t) < \infty$ for any $t \in T$; (2) σ-finite if $\mathbf{n} = \varkappa_0$; (3) totally additive if for any system of mutally orthogonal subspaces, $\{N_a: a \in A\}$, $m(\bigoplus_{a \in A} N_a) = \sum_{a \in A} m(N_a)$; (4) a finite Gleason measure if it is expressible via (1).

Theorem 1 (Eilers and Horst). Any finite measure m on $\mathscr{L}(H)$, dim $H \neq 2$, is a finite Gleason measure iff the dimension of H is a nonmeasurable cardinal.

Proof. Assume that any finite measure on $\mathscr{L}(H)$ is expressible via (1). We claim to show that the dimension of H is a nonmeasurable cardinal. Suppose the converse. Let $\{e_t: t \in T\}$ be an orthonormal basis in H. Choose a probability measure ν on 2^T vanishing at any one-point subset of T.

A map m on $\mathscr{L}(H)$ defined by

$$m(M) = \int_T \|P^M e_t\|^2 \, d\nu(t), \qquad M \in \mathscr{L}(H) \tag{2}$$

is a finite measure on $\mathscr{L}(H)$ with $m(H) = 1$ and $m(P_{e_t}) = 0$ for every $t \in T$, where by P_f we mean an one-dimensional subspace of H spanned over a nonzero vector $f \in H$.

Maeda (1980) proved that any finite measure on $\mathscr{L}(H)$ of an arbitrary Hilbert space H, dim $H \neq 2$, is expressible via (1) iff it is totally additive. Hence, a measure m defined by (2) is totally additive. Consequently,

$$1 = m(H) = m(\bigoplus_{t \in T} P_{e_t}) = \sum_{t \in T} m(P_{et}) = 0$$

where is a contradiction. Therefore, the dimension of H is a nonmeasurable cardinal.

The converse implication has been proved by Eilers and Horst (1975) [see also Drisch (1979)]. ■

In the following result we generalize Theorem 1 to σ-finite measures attaining infinite values. For this reason we need further notations. Let t be a symmetric bilinear form with a dense domain $D(t)$, that is: (1) $D(t)$ is a linear submanifold; (2) t: $D(t) \times D(t) \to C$ (C is the field of scalars) such that t is linear in the first argument and antilinear in the second one; (3) $t(\alpha x, \beta y) = \alpha\bar{\beta} t(x, y)$ for all $x, y \in D(t)$ and all $\alpha, \beta \in C$; (4) $t(x, y) = \overline{t(y, x)}$ for all $x, y \in D(t)$. Let $M \in \mathscr{L}(H)$ and let $M \subset D(t)$. Then by $t \circ M$ we mean a symmetric bilinear form defined by $t \circ M(x, y) = t(P^M x, P^M y)$, $x, y \in H$. If $t \circ M$ is induced by a trace operator T, such that $t \circ M(x, y) = (Tx, y)$, $x, y \in H$, then we say $t \circ M \in \mathrm{Tr}(H)$ and we put tr $t \circ M =$ tr T, where $\mathrm{Tr}(H)$ is the set of all trace operators in H. A bilinear form t is positive if $t(x, x) \geq 0$ for any $x \in D(t)$.

Lugovaja and Sherstnev (1980) proved that for any σ-finite measure m on $\mathscr{L}(H)$, $m(H) = \infty$, of a Hilbert space H with dim $H = \aleph_0$, there exists a unique positive symmetric bilinear form t with a dense domain such that

$$m(M)=\begin{cases} \operatorname{tr} t\circ M & \text{iff} \quad t\circ M\in \operatorname{Tr}(H) \\ \infty & \text{otherwise} \end{cases} \tag{3}$$

Since (3) generalizes the formula (1), any measure m for which there is a positive symmetric bilinear form t with a dense domain such that (3) holds is said to be a Gleason measure.

Theorem 2. Any σ-finite measure m on $\mathscr{L}(H)$, $m(H)=\infty$, $\dim H\neq 2$, is a Gleason measure iff the dimension of H is a nonmeasurable cardinal.

Proof. If is evident that any cardinal I is nonmeasurable iff there is no σ-finite measure ν on 2^T such that $\nu(T)=\infty$, $\nu(\{t\})=0$ for any $t\in T$ and card $T=I$.

Suppose that any σ-finite measure on $\mathscr{L}(H)$ is a Gleason one and let the dimension of H not be a nonmeasurable cardinal. Analogous as in the proof of Theorem 1, we define a map m by (2), where now ν is a σ-finite measure on 2^T with $\nu(T)=\infty$ and vanishing at any one-point subset of T. This m is a σ-finite measure and, due to the assumption, it is a Gleason one.

From Dvurečenskij (1986) it follows that any σ-finite measure on an arbitrary quantum logic $\mathscr{L}(H)$, $\dim H\neq 2$, is a Gleason one iff it is totally additive. This gives the contradiction.

The converse implication follows from Theorem 6 of Dvurečenskij (1986). ■

Finally we note an interesting fact which has a connection with non-measurable cardinals. It is known, due to Dvurečenskij (1987), that there are a Hilbert space H of infinite nonmeasurable cardinality and a positive symmetric bilinear form t with a dense domain such that t determines via (3) a Gleason measure m with $m(H)=\infty$ and which is not $\mathbf{n}$-finite for any cardinal $\mathbf{n}$.

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